Criterion to identify Hopf bifurcations in maps of arbitrary dimension

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The classical Hopf bifurcation criterion is stated in terms of the properties of eigenvalues. In this paper, a criterion without using eigenvalues is proposed for maps of arbitrary dimension. The parameter mechanism of Hopf bifurcation may be explicitly formulated on the basis of the criterion. A numerical example demonstrates that the proposed criterion is preferable to the classical Hopf bifurcation criterion in theoretical analysis and practical applications.

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Hopf bifurcation is a common phenomenon in physical, chemical, biological, electronic, and other engineering fields [1]. The criteria of Hopf bifurcation for maps consist of the eigenvalue assignment, the transversality condition, and the nonresonance (or resonance) condition [2]. The two former conditions determine the existence of a Hopf bifurcation, and the last is involved in the type of bifurcation solutions. The classical Hopf bifurcation criterion [2] is repeated as follows.

Consider an n-dimensional map $x_{k+1}=f_\mu(x_k)$ where $x_{k+1}, x_k \in R^n$, are the state vectors, k is iterative index, and $\mu \in R$ is a parameter. Assume that f_μ has a fixed point x_0 and satisfies the following.

(C1) *Eigenvalue assignment*. The Jacobian matrix $D_{x_k} f_\mu(x_0)$ has a pair of complex conjugate eigenvalues $\lambda_1(\mu)$ and $\bar{\lambda}_1(\mu)$ with $|\lambda_1(\mu_0)|=1$ at $\mu=\mu_0$ and the others $\lambda_j(\mu)$, $j=3,\ldots,n$, with $|\lambda_j(\mu_0)|<1$.

(C2) *Transversality condition* $d|\lambda_1(\mu_0)|/d\mu \neq 0$.

(C3) *Nonresonance condition* $\lambda_1^m(\mu_0)\neq 1$ or *resonance condition* $\lambda_1^m(\mu_0)=1$, $m=3,4,5,\ldots$.

Then, a Hopf bifurcation occurs at $\mu=\mu_0$. The type and stability of bifurcation solutions depend on the condition (C3) and the nonlinear property of map f_μ.

Detecting the existence of a Hopf bifurcation is one of the oldest nonlinear topics that remain prevalent. The main idea in the literature is to directly compute all eigenvalues of the Jacobian matrix $D_{x_k} f_\mu(x_0)$ and check the classical criterion (C1)–(C3) which are stated in terms of the properties of eigenvalues. In each step of computations, the parameter μ is preset or speculated such that the Jacobian matrix $D_{x_k} f_\mu(x_0)$ becomes a constant matrix. In this case, the eigenvalues of the constant matrix $D_{x_k} f_\mu(x_0)$ are computable in general. However, for practical physical systems, $D_{x_k} f_\mu(x_0)$ may involve certain singularities such as a sparse matrix, which introduce numeric inaccuracies into eigenvalue computations.

In practical engineering studies of the dynamic behavior of physical systems, it is often desirable to reveal the multiparameter mechanism of the bifurcation. Subject to the eigenvalue computation point by point in the parameter plane or hyperplane, it is very difficult to apply the criterion (C1)–(C3) to serve this purpose. For example, Chen and coworkers [3] presented the seminal work on the creation of a Hopf bifurcation with certain desired dynamical properties via control. In contrast to the studies of detecting a Hopf bifurcation, the remarkable property of this new topic is that the Jacobian matrix $D_{x_k} f_\mu(x_0)$ always involves a control parameter vector K (as usual, the higher dimensional of the system, the more components of K) to be determined such that the matrix $D_{x_k} f_\mu(x_0;K)$ is not a constant matrix. Furthermore, analytical expressions for all eigenvalues with respect to K or μ, in general, are unavailable for a high dimensional nonconstant matrix. These properties make it a nontrivial task to design the Hopf bifurcation depending on the criterion (C1)–(C3).

In this paper, a criterion of Hopf bifurcation is proposed for any map in a general sense. Without using eigenvalues, the criterion is formulated using a set of simple equalities or inequalities that consist of the coefficients of the characteristic equation derived from the Jacobian matrix. In a comparison of the classical Hopf bifurcation criterion, the newly derived criterion is more efficient in detecting the existence of Hopf bifurcation for high dimensional maps as the result of eliminating singularities when computing eigenvalues. Moreover, even if the Jacobian matrix involves some unknown parameters, the relationship between the unknown parameters and the critical bifurcation constraint conditions is explicitly expressed. This property completely overcomes the difficulty in the existing results [3–5] of the creation of Hopf bifurcations via control.

In order to express the criterion, assume first that at the fixed point x_0 the characteristic polynomial of an n-dimensional map f_μ takes the form

$$p_\mu(\lambda)=\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n, \tag{1}$$

where $a_j=a_j(\mu,K)$, $j=1,\ldots,n$, μ is the bifurcation parameter, and K is the control parameter or the other to be determined. Consider the sequence of determinants $\Delta_0^\pm(\mu,K)=1$, $\Delta_1^\pm(\mu,K),\ldots,\Delta_n^\pm(\mu,K)$, where

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$$\Delta_j^{\pm}(\mu,K)=\left|\begin{pmatrix}1 & a_1 & a_2 & \cdots & a_{j-1}\\ 0 & 1 & a_1 & \cdots & a_{j-2}\\ 0 & 0 & 1 & \cdots & a_{j-3}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 1\end{pmatrix}\pm\begin{pmatrix}a_{n-j+1} & a_{n-j+2} & \cdots & a_{n-1} & a_n\\ a_{n-j+2} & a_{n-j+3} & \cdots & a_n & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{n-1} & a_n & \cdots & 0 & 0\\ a_n & 0 & \cdots & 0 & 0\end{pmatrix}\right|,\quad j=1,\ldots,n. \tag{2}$$

Now the following conditions (H1)–(H3) can be formulated to establish a criterion of Hopf bifurcation.

(H1) *Eigenvalue assignment* $\Delta_{n-1}^{-}(\mu_0,K)=0$, $p_{\mu_0}(1)>0$, $(-1)^n p_{\mu_0}(-1)>0$, $\Delta_{n-1}^{+}(\mu_0,K)>0$, $\Delta_j^{\pm}(\mu_0,K)>0$, $j=n-3,n-5,\ldots,1$ (or 2), when n is even (or odd, respectively).

(H2) *Transversality condition* $d\Delta_{n-1}^{-}(\mu_0,K)/d\mu\neq 0$.

(H3) *Nonresonance condition* $\cos(2\pi/m)\neq\psi$ or *resonance condition* $\cos(2\pi/m)=\psi$, where $m=3,4,5,\ldots$ and $\psi=1-0.5p_{\mu_0}(1)\Delta_{n-3}^{-}(\mu_0,K)/\Delta_{n-2}^{+}(\mu_0,K)$.

For map f_μ, if (H1)–(H3) hold, then Hopf bifurcation occurs at μ_0.

Proof. If the conditions (H1)–(H3) are equivalent to the conditions (C1)–(C3), then they are a criterion of Hopf bifurcation for maps. At first, we highlight three steps to show that (H1) yields (C1). We also verify that there exists one and only pair of eigenvalues that are inverse with respect to the unit circle, show that the sole inverse pair lies on the unit circle and is complex conjugate, and ascertain that the other eigenvalues lie inside the unit circle. It should be noted that the determinant $\Delta_{n-1}^{-}(\mu,K)$ holds the following result [6]:

$$\Delta_{n-1}^{-}(\mu,K)=(-1)^{n(n-1)/2}\prod_{j<m}^{1,\ldots,n}(1-\lambda_j\lambda_m), \tag{3}$$

where λ_j and λ_m are the roots of $p_\mu(\lambda)=0$. Thus, $\Delta_{n-1}^{-}(\mu_0,K)=0$ guarantees at least a pair of eigenvalues that are on the unit circle or inverse points with respect to the unit circle, which are denoted by h and $1/h$. We then rewrite $p_{\mu_0}(\lambda)$ as

$$p_{\mu_0}(\lambda)=(\lambda^2-(h+1/h)\lambda+1)\tilde{p}_{\mu_0}(\lambda), \tag{4}$$

where $\tilde{p}_{\mu_0}(\lambda)=\lambda^{n-2}+b_1\lambda^{n-3}+\cdots+b_{n-3}\lambda+b_{n-2}$. For the coefficients b_m and a_m in Eq. (1), one obtains the relationship

$$a_m=b_m-(h+1/h)b_{m-1}+b_{m-2}, \tag{5}$$

where $m=1,\ldots,n$, $b_0=1$, and $b_j=0$ if $j>(n-2)$ or $j<0$. A set of determinants $\tilde{\Delta}_j^{\pm}(\mu_0,K)$, $j=n-3,n-5,\ldots,1$ or 2, of $\tilde{p}_\mu(\lambda)$ is defined similar to Eq. (2). One then substitutes Eq. (5) into $\Delta_j^{\pm}(\mu_0,K)$, $j=n-3,n-5,\ldots,1$ or 2, and makes the elementary row operations for each row of $\Delta_j^{\pm}(\mu_0,K)$ as follows: starting from the last row of $\Delta_j^{\pm}(\mu_0,K)$, multiply the mth row by $(h+1/h)$ and -1, and add them to the $(m-1)$th and $(m-2)$th rows (if any), respectively, to obtain

$$\Delta_j^{\pm}(\mu_0,K)=\tilde{\Delta}_j^{\pm}(\mu_0,K),\quad j=n-3,n-5,\ldots,1\text{ or }2. \tag{6}$$

From (H1) and Eq. (6), one can easily obtain $\tilde{\Delta}_{n-3}^{-}(\mu_0,K)>0$. Furthermore, the relationship (3) also holds for $\tilde{\Delta}_{n-3}^{-}(\mu_0,K)$ with a minor modification. Thus, $\tilde{p}_{\mu_0}(\lambda)$ has no inverse eigenvalues with respect to the unit circle. In other words, h and $1/h$ are the sole pair of inverse eigenvalues for $p_{\mu_0}(\lambda)$. If either h or $1/h$ is outside the unit circle, then at least one determinant among $\Delta_j^{\pm}(\mu_0,K)$, $j=n-1,n-3,\ldots,1$ or 2, is negative [7,8]. But this contradicts our assumption (H1). One thus can ascertain that both h and $1/h$ are on the unit circle. In view of $p_{\mu_0}(1)>0$ and $(-1)^n p_{\mu_0}(-1)>0$, it is clear that $p_{\mu_0}(\lambda)$ has no real eigenvalue on the unit circle. This implies that h and $1/h$ are a pair of complex conjugate eigenvalues—that is, $h=\alpha+i\beta$ and $1/h=\bar{h}$ with $\alpha^2+\beta^2=1$ and $|\alpha|<1$. In what follows, we show that the other eigenvalues are inside the unit cycle. The necessary and sufficient condition [7,9] that the roots of $\tilde{p}_{\mu_0}(\lambda)=0$ lie inside the unit circle is given by

$$\tilde{p}_{\mu_0}(1)>0,\quad(-1)^{n-2}\tilde{p}_{\mu_0}(-1)>0,\quad\tilde{\Delta}_j^{\pm}(\mu_0,K)>0,$$
$$j=n-3,n-5,\ldots,1\text{ or }2. \tag{7}$$

Since $|(h+1/h)|=|2\alpha|<2$, one can see from Eq. (4) that the signs of $\tilde{p}_{\mu_0}(1)$ and $(-1)^{n-2}\tilde{p}_{\mu_0}(-1)$ are the same as those of $p_{\mu_0}(1)$ and $(-1)^n p_{\mu_0}(-1)$, respectively. Furthermore, substituting Eq. (6) into (H1), it follows from Eq. (7) that all roots of $\tilde{p}_{\mu_0}(\lambda)=0$ lie inside the unit circle. Therefore, the condition (H1) leads to (C1).

If the condition (C1) holds, $p_{\mu_0}(\lambda)$ can be expressed as Eq. (4) where $(h+1/h)=\alpha$ and $|\alpha|<1$. By applying the above procedures, inversely, one can show that the condition (C1) yields (H1) except for $\Delta_{n-1}^{+}(\mu_0,K)>0$. For example, by substituting Eq. (5) into $\Delta_{n-1}^{-}(\mu_0,K)$ and making the elementary row operations in the derivation of Eq. (6), the first and third rows of $\Delta_{n-1}^{-}(\mu_0,K)$ become equal or else the components of the first row all are zero such that

$$\Delta_{n-1}^{-}(\mu_0,K)=0. \tag{8}$$

In order to verify $\Delta_{n-1}^{+}(\mu_0,K)>0$, as above, one obtains

$$\Delta_{n-2}^{\pm}(\mu_0,K)=\tilde{\Delta}_{n-2}^{\pm}(\mu_0,K). \tag{9}$$

The following three formulations proven in [7] are needed to serve our purpose:

$$\tilde{\Delta}^{+}_{n-2}(\mu_0,K) = \tilde{p}_{\mu_0}(1)\tilde{\Delta}^{-}_{n-3}(\mu_0,K),$$
$$\tilde{\Delta}^{-}_{n-2}(\mu_0,K) = (-1)^{n-2}\tilde{p}_{\mu_0}(-1)\tilde{\Delta}^{-}_{n-3}(\mu_0,K) \tag{10}$$

and

$$\Delta^{+}_{n-3}(\mu_0,K)\Delta^{-}_{n-1}(\mu_0,K) + \Delta^{-}_{n-3}(\mu_0,K)\Delta^{+}_{n-1}(\mu_0,K) = 2\Delta^{+}_{n-2}(\mu_0,K)\Delta^{-}_{n-2}(\mu_0,K). \tag{11}$$

Substituting Eqs. (6)–(10) into Eq. (11), $\Delta^{+}_{n-1}(\mu_0,K)>0$ is readily ascertained. Therefore, the condition (H1) is equivalent to (C1).

Next, by differentiating $\Delta^{-}_{n-1}(\mu,K)$ in Eq. (3) with respect to μ, we have

$$d\Delta^{-}_{n-1}(\mu,K)/d\mu = (-1)^{n(n-1)/2}\sum_{l<q}^{1,\ldots,n}\left(\lambda_l'\lambda_q/(\lambda_l\lambda_q-1)\times\prod_{j<m}^{1,\ldots,n}(1-\lambda_j\lambda_m) + \lambda_l\lambda_q'/(\lambda_l\lambda_q-1)\times\prod_{j<m}^{1,\ldots,n}(1-\lambda_j\lambda_m)\right), \tag{12}$$

where λ_l' denotes the derivative of λ_l with respect to μ, so is λ_q'. The equivalence of (H1) to (C1) implies that at μ_0 the characteristic equation $p_{\mu_0}(\lambda)=0$ has one and only pair of complex conjugate eigenvalues λ_1 and $\lambda_2=\bar{\lambda}_1$ on the unit circle, satisfying $(\lambda_1\lambda_2-1)=0$, such that $\prod_{j<m}^{1\ldots n}(1-\lambda_j\lambda_m)/(\lambda_1\lambda_2-1)\neq 0$. Note that the sum of the right-hand side of Eq. (12) has one and only term without $(\lambda_1\lambda_2-1)$. Thus we get

$$d\Delta^{-}_{n-1}(\mu_0,K)/d\mu = 2(-1)^{n(n-1)/2}d|\lambda_1(\mu_0)|/d\mu \times\prod_{j<m}^{1,\ldots,n}(1-\lambda_j\lambda_m)/(\lambda_1\lambda_2-1). \tag{13}$$

From the preceding identity, one can ascertain that the condition (H2) is equivalent to (C2).

Finally, a tactical method is used to show the equivalence of (H3) to (C3). Assume that $\lambda_1(\mu_0)=\alpha+i\beta$ and $|\lambda_1(\mu_0)|=1$. Note that $\lambda_1^m(\mu_0)\neq 1$ [respectively, $\lambda_1^m(\mu_0)=1$] corresponds to $\alpha\neq\cos(2\pi/m)$ [respectively, $\alpha=\cos(2\pi/m)$] owing to $|\lambda_1(\mu_0)|=1$. Furthermore, it is seen from $p_{\mu_0}(\lambda)=(\lambda^2-2\alpha\lambda+1)\tilde{p}_{\mu_0}(\lambda)$ that $\alpha=1-0.5p_{\mu_0}(1)/\tilde{p}_{\mu_0}(1)$. Substituting Eqs. (6) and (9) into Eq. (10), one can easily obtain that $\tilde{p}_{\mu_0}(1)=\Delta^{+}_{n-2}(\mu_0,K)/\Delta^{-}_{n-3}(\mu_0,K)$. Therefore, (H3) is equivalent to (C3). ■

Note that the proposed criterion is applicable for any map in a general sense, such as the Poincaré map of impact vibrators [10], the A-switching map describing dc/dc converters [10], a four-dimensional revisable map [10], the three-order Rodriguez-Vazquez map [10], the well-known bounce ball map [2], and coupled map lattices [11]. Grassi and Miller [12] studied the electronic implement of the generalized Hénon map by analog circuits and applied it to a chaos synchronization experiment. To show that the proposed criterion is very helpful in the investigation of the parameter mechanism of the bifurcation, consider the following parameter matrix:

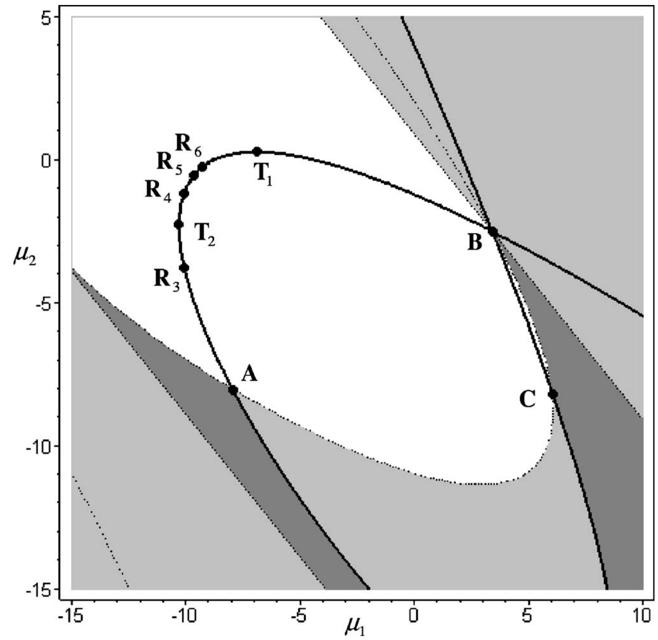

FIG. 1. The open arcs AB and BC (the heavy solid lines) in the blank region, except for the points T_1 and T_2, represent the critical parameter set of Hopf bifurcation. The points R_3, R_4, R_5, and R_6 represent the resonance points with $\lambda_1^m(\mu_0)=1$, $m=3,4,5,6$, respectively. The open blank region surrounded with the solid arcs AB and BC and the dotted arc CA is the stable parameter domain of the fixed point x_0.

$$D_{x_k}f_\mu(x_0) = \begin{pmatrix} 0 & -0.5 & -0.9 & -0.1 & 0.8 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \mu_1 & 1 & 0 & \mu_2 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}, \tag{14}$$

where $\mu_1\in[-15,10]$ and $\mu_2\in[-15,5]$ are two uncertain parameters or control parameters. The matrix (14) is the Jacobian one of the generalized Hénon map under washout-filter-aided control [12], with a minor modification for simplicity. The conditions (H1)–(H3) are described in Fig. 1. The blank region denotes the parameter domain in which all inequalities in (H1) hold whereas in the gray region at least one inequality fails. In the open domain surrounded with the solid arcs AB and BC and the dotted arc CA, one has $\Delta^{-}_{n-1}(\mu_0,K)>0$ such that the fixed point x_0 is local stable according to the stability criterion (7). The open arcs AB and BC (the heavy solid lines) consist of the parameter points that satisfy $\Delta^{-}_{n-1}(\mu_0,K)=0$ as well as the inequality constraints in (H1). Except for the points T_1 and T_2, they represent the critical parameter set of the Hopf bifurcation. The point T_1 (respectively, T_2) refers to the one at which (H2) fails for μ_1 (respectively, μ_2). At point B, (H2) fails for both μ_1 and μ_2. The loss of stability of the fixed point x_0 caused by Hopf bifurcation is not sensitive to the parameters that are very close to these points. When our objective is to design Hopf bifurcation into the system, an appreciate parameter setting should be away from these points. The points R_3, R_4, R_5, and R_6 represent the resonance points with $\lambda_1^m(\mu_0)=1$, $m=3,4,5,6$, respectively. To design Hopf bifurcation in the nonresonance cases, the bifurcation parameters should be on the open arcs AB and BC but be away from these resonance points. If one attempts to design Hopf bifurcation in the case of a kind of resonance, one of these resonance points is exactly required in the choice of parameters. The mechanism of the uncertain parameters in Hopf bifurcations is clearly shown in Fig. 1. However, based on the classical Hopf bifurcation criterion described by (C1)–(C3), it is a nontrivial task to systematically demonstrate the information. For example, depending on eigenvalue computation by scanning the parameter space point by point, we have to keep our fingers crossed to find the open arcs AB and BC that satisfy assumption (C1).

In summary, a Hopf bifurcation criterion without using eigenvalues has been proposed. By applying it, the parameter mechanism of Hopf bifurcation may be explicitly formulated. The criterion is more convenient and efficient for analyzing Hopf bifurcations than the classical Hopf bifurcation criterion, especially for high dimensional maps with uncertain parameters.

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